

Linking Classical and Quantum Stochastic Processes

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We define a map which relates four dimensional classical stochastic matrices to qubit quantum channels. The map preserves the spectrum and the composition of processes. To do this we introduce the concept of Bloch tetrahedron which plays the same role in the classical context as the Bloch sphere in the quantum context. A similar map is also induced between dynamical generators of classical and quantum stochastic processes. Possibilities for generalization to arbitrary dimensions are also discussed.

The state of an open physical system interacting with an environment on which we have little or no control, is given by a probability vector $|P\rangle$ in the classical domain or a density matrix ρ in quantum domain. Linear evolution equations describe the dynamics of such states in successive time steps. In the classical case, the dynamics is given by $|P_{n+1}\rangle = Q|P_n\rangle$ [1] where n denotes the time step and the stochastic matrix Q collects the jumps probabilities. In the quantum domain, the dynamics is represented by a completely positive trace-preserving (CPT) map or quantum channel $\rho_{n+1} = \mathcal{E}(\rho_n)$ [2]. These equations provide an abstract but very general framework in which the dynamics of many different phenomena both in classical and quantum physics can be analyzed.

In classical domain, this might describe a random walk by a particle, diffusion and reaction of particles, successive conformation of a polymer chain, radioactive decay of nuclei, or population dynamics of any system of interacting entities. Its application even goes beyond physics to other fields of science, like biology and finance [3].

In quantum domain such an equation may describe an application of a gate in a quantum processor, a communication channel in time (quantum storage device) or space (an optical fiber), or any open systems dynamics where one merely has partial access to the relevant degrees of freedom [4].

How quantum stochastic processes are related to classical ones? Is it possible to construct a quantum channel for every stochastic matrix? How about the converse? If yes, how the properties of the classical stochastic process, are reflected in the quantum channel? What does it imply about the true quantum nature of a quantum channel? What types of quantum stochastic processes can be simulated by classical computers? What are the precise obstacles for establishing a one-to-one correspondence between these two domains, at least in those cases where we have a complete characterization of quantum channels, like the qubit channels? In this letter we provide results which shed light on these questions. The point of view adapted in this letter, complements many other attempts [5] for understanding various aspects of open quantum system dynamics. Specifically we show that

a) Four dimensional probability vectors can be characterized in complete analogy with qubit density matrices, where here Bloch tetrahedron replaces Bloch sphere,

b) four dimensional stochastic matrices can be characterized in almost complete analogy with qubit channels,

c) from these two results we make a map between four dimensional stochastic matrices and qubit channels, a map which relates many of the properties, i.e. spectrum, divisibility etc. of one domain to the other.

d) We also show that the map can be extended to relate d^2 dimensional stochastic matrices to d -dimensional quantum channels. This map can play a useful role in characterization of quantum channels in arbitrary dimension, where a complete characterization is missing.

It is worth noting that we go beyond the existing relation [6] between d dimensional stochastic matrices and the restriction of quantum channels on d -dimensional diagonal density matrices (In these cases, the image of a stochastic matrix is always a zero-determinant channel.) We show that the natural relation is between d^2 probability vectors and d -dimensional density matrices, a relation which is highly nontrivial and very reach indeed. We start with a detailed exposition in the $d = 2$ case.

Consider a system with four configurations. By P_μ , $\mu = 0, 1, 2, 3$, we denote the probability of the system to be in each of these configurations. The state of the system is described by the vector $|P\rangle = (P_0 \ P_1 \ P_2 \ P_3)^T$, where T stands for transpose. Obviously $0 \leq P_\mu \leq 1$ and $\sum_\mu P_\mu = 1$. So there are three independent parameters in $|P\rangle$. The constraint $\sum_{\mu=0}^3 p_\mu = 1$ is satisfied if we re-parameterize $|P\rangle$ in the following form

$$|P\rangle = \frac{1}{4} \begin{pmatrix} 1 + r_1 + r_2 + r_3 \\ 1 + r_1 - r_2 - r_3 \\ 1 - r_1 + r_2 - r_3 \\ 1 - r_1 - r_2 + r_3 \end{pmatrix}, \quad (1)$$

where r_i are three real numbers, the domain of which will be determined later. The components of $|P\rangle$ can be

written as

$$P_\mu = \frac{1}{4}(1 + e_\mu \cdot r), \quad (2)$$

where the four 3-dimensional vectors e_μ ($\mu = 0, 1, 2, 3$)

$$\begin{aligned} e_0 &:= (1, 1, 1) & e_1 &:= (1, -1, -1) \\ e_2 &:= (-1, 1, -1) & e_3 &:= (-1, -1, 1), \end{aligned} \quad (3)$$

point to the four corners of a tetrahedron which we call the Bloch Tetrahedron and denote it by Δ , figure (1). These vectors have the following inner products,

$$e_\mu \cdot e_\nu = 4\delta_{\mu,\nu} - 1. \quad (4)$$

Moreover they satisfy

$$\sum_\mu e_\mu = 0, \quad \sum_\mu (e_\mu)_i (e_\mu)_j = 4\delta_{ij}. \quad (5)$$

From these 3-dimensional vectors we can construct an orthonormal basis for the four dimensional space

$$|e_\mu\rangle = \frac{1}{2}(1, e_\mu)^T, \quad (6)$$

and write $|P\rangle$ as

$$|P\rangle = \frac{1}{2}(|e_0\rangle + r_1|e_1\rangle + r_2|e_2\rangle + r_3|e_3\rangle). \quad (7)$$

In order that $|P\rangle$ be a valid probability vector, all its entries should be non-negative and less than unity. This yields the following conditions for the vector r ,

$$-1 \leq e_\mu \cdot r \leq 3, \quad (8)$$

which means that the vector r , should lie in the Bloch tetrahedron Δ . In fact Δ plays the same role for classical probability vectors as Bloch sphere plays for density matrices. There is a one-to-one correspondence between classical probability vectors and points in the Bloch tetrahedron. The point $r = 0$ corresponds to the completely mixed state, where all configurations are equally probable. The corners of the tetrahedron correspond to pure states, where only one configuration has non-zero probability.

The basis $\{|e_\mu\rangle\}$ allows us to characterize a four dimensional stochastic matrix in an elegant way, which is structurally very similar to the characterization of qubit channels, albeit with important differences which will be discussed later on. A stochastic matrix Q has the property that the entries in each column sum up to 1, that is $\langle e_0|Q = \langle e_0|$, hence the expansion

$$Q = |e_0\rangle\langle e_0| + t_i|e_i\rangle\langle e_0| + \Lambda_{ij}|e_i\rangle\langle e_j|, \quad (9)$$

where a summation over repeated indices is understood. Note that Greek indices run from 0 to 3 and the Latin indices from 1 to 3. We denote it by $Q_{t,\Lambda}$, where t is a three

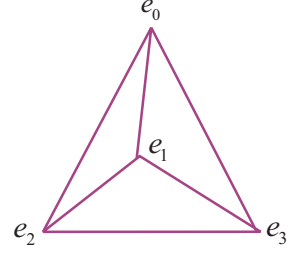


FIG. 1: Bloch Tetrahedron: Every vector in this tetrahedron corresponds to a probability vector according to (2). The corners correspond to pure states and the center to the completely mixed state.

dimensional vector and Λ is a real three-dimensional square matrix. When acting on a probability vector, this matrix induced an affine transformation on the vector r , that is,

$$Q_{t,\Lambda} : |P(r)\rangle \longrightarrow |P(\Lambda r + t)\rangle. \quad (10)$$

Therefore $Q_{t,\Lambda}$ will be a stochastic matrix provided that the affine transformation $r \longrightarrow r' := \Lambda r + t$ maps Bloch tetrahedron to Bloch tetrahedron (We remind the reader of a similar requirement of positive maps for which the affine transformation should map Bloch sphere to Bloch sphere). This puts severe constraint on the matrix Λ and the translation t . For simplicity, hereafter we restrict ourselves to doubly stochastic matrices, for which $t = 0$. In order to find the conditions on the matrix Λ , we first try to put $Q_{0,\Lambda}$ into a normal form.

Normal form of Q : Consider a doubly stochastic matrix $Q_{0,\Lambda}$. We can make a singular value decomposition of Λ as

$$\Lambda = \hat{S}\Lambda_D\hat{T}, \quad (11)$$

where the diagonal matrix $\Lambda_D = \text{diagonal}(\lambda_1, \lambda_2, \lambda_3)$ contains singular values of Λ . \hat{S} and \hat{T} are three dimensional orthogonal matrices and their embedding in four dimensions, in the basis $\{|e_\mu\rangle\}$ is of the form

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \hat{S} \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \hat{T} \end{pmatrix}. \quad (12)$$

We will then have

$$Q_{0,\Lambda} = SQ_{0,\Lambda_D}T, \quad (13)$$

where $Q_n := Q_{0,\Lambda_D}$ in the right hand side is called the normal form of the stochastic matrix Q and has the following expression

$$Q_n := |e_0\rangle\langle e_0| + \lambda_i|e_i\rangle\langle e_i|. \quad (14)$$

Let us now ask under what conditions, the normal form Q_n is a stochastic matrix. This is investigated in the next theorem.

Theorem 1. A matrix Q_n is a doubly stochastic matrix if and only if the vector $\lambda := (\lambda_1, \lambda_2, \lambda_3)$ belongs to Δ .

Proof. Using (6), we find that the explicit form of the matrix elements of Q_n . They are all of the form $\frac{1}{4}(1 + e_\mu \cdot \lambda)$ for some e_μ . Therefore the elements of Q_n are positive if and only if $-1 \leq \lambda \cdot e_\mu \leq 3$, $\forall \mu$, which means that $\lambda \in \Delta$. \square

The above representation, as we will see later on, is very apt for constructing a map between classical four dimensional stochastic matrices and qubit channels. To this end consider the following two-dimensional matrices

$$A_\mu := \frac{1}{2}(I + e_\mu \cdot \sigma), \quad \mu = 0, 1, 2, 3. \quad (15)$$

These matrices are Hermitian, have unit trace and form a basis for the space of 2×2 matrices, but are not positive. Moreover they satisfy

$$\text{tr}(A_\mu A_\nu) = 2\delta_{\mu,\nu}, \quad \sum_\mu A_\mu = 2I. \quad (16)$$

Definition 1. Consider now the following map, $Q \rightarrow \mathcal{E}_Q$, which takes a matrix Q to a qubit map

$$\mathcal{E}_Q(\rho) := \frac{1}{2} \sum_{\mu,\nu} Q_{\mu,\nu} \text{tr}(\rho A_\nu) A_\mu. \quad (17)$$

This map has the following properties:

i) It respects composition, that is

$$\mathcal{E}_{Q_1 Q_2} = \mathcal{E}_{Q_1} \mathcal{E}_{Q_2}, \quad (18)$$

ii) It maps an orthogonal matrix like S to a unitary channel, i.e.

$$\mathcal{E}_S(\rho) = U^{-1} \rho U, \quad (19)$$

where U is the representation of the orthogonal matrix S on the basis A_μ , i.e. $U A_\mu U^{-1} = S_{\mu,\nu} A_\nu$,

iii) If $\sum_{\mu=0} Q_{\mu,\nu} = 1$ ($\sum_{\nu=0} Q_{\mu,\nu} = 1$), then \mathcal{E}_Q is trace-preserving (unital), finally

iv) It preserves the spectrum, that is $Q|v\rangle = v|v\rangle$, if and only if $\mathcal{E}_Q(X_v) = vX_v$, where $X_v := \sum_\alpha v_\alpha A_\alpha$.

From (i), (ii) and the decomposition $Q = S Q_n T$, we find that

$$\mathcal{E}_Q(\rho) = U^{-1} \mathcal{E}_{Q_n}(V^{-1} \rho V) U. \quad (20)$$

Note that V is defined similarly to U , i.e. $V A_\mu V^{-1} = T_{\mu,\nu} A_\nu$. Equation (20) shows the standard decomposition of any CPT map for qubits [7]. Using (15) and (17), the explicit form of \mathcal{E}_{Q_n} [8] turns out to be

$$\mathcal{E}_{Q_n}(\rho) = \frac{1}{2}(\text{tr}(\rho)I + \lambda_i \text{tr}(\rho \sigma_i) \sigma_i) \quad (21)$$

Despite the non-positivity of the matrices $\{A_\mu\}$ we have the following

Theorem 2. The map \mathcal{E}_{Q_n} is a unital quantum channel, if only if Q_n is a doubly stochastic matrix.

Proof. The Choi-Jamiolkowski [9] matrix of \mathcal{E}_{Q_n} is found to be

$$\tau(\mathcal{E}_{Q_n}) = \frac{1}{4}(I \otimes I + \lambda_i \sigma_i \otimes \sigma_i^T). \quad (22)$$

The eigenvalues of this matrix can easily be calculated and can be expressed as

$$\frac{1}{4}(1 + \lambda \cdot e_\mu), \quad \mu = 0, 1, 2, 3. \quad (23)$$

Therefore we have arrived at the nice result that the eigenvalues of the Choi-Jamiolkowski matrix of \mathcal{E}_{Q_n} are exactly the matrix elements of the doubly stochastic matrix Q_n . Hence, these eigenvalues are non-negative if and only if the entries of Q_n are non-negative. Therefore Q_n is a doubly stochastic matrix, if and only if \mathcal{E}_{Q_n} is a completely positive unital map. \square

Moreover in view of (20) we find that the map \mathcal{E}_Q corresponding to $Q \equiv S Q_n T$, is also CPT and unital, even though for some parameters, the matrix $S Q_n T$ may no longer be stochastic, since S and T are not necessarily stochastic.

As a very simple example, the classical depolarizing matrix $Q = p I + (1-p)|e_0\rangle\langle e_0|$ is already in normal form and is mapped to the quantum depolarizing channel $\mathcal{E}_Q(\rho) = p\rho + (1-p)\frac{I}{2}$.

The above results suggest that we may also be able to map the generators of classical stochastic matrices in four dimensions to Lindblad generators [10, 11] of Markovian quantum evolution for qubits. This is indeed the case, provided that the matrix generates symmetric stochastic processes, i.e. $Q_{\mu,\nu} = Q_{\nu,\mu}$.

Let H be such a symmetric generator which, due to the condition $\langle e_0|H = 0$, will have the following expression in the basis (6)

$$H = \sum_{i,j} H_{ij} |e_i\rangle\langle e_j|, \quad (24)$$

where $H_{ij} = H_{ji}$. In the computational or more aptly the configurational basis, the off-diagonal elements of such a matrix, representing the rates of jumps between different configurations are non-negative and consequently the diagonal entries are negative. It also has a normal form as $H = S' H_n S'^T$, where

$$H_n = \sum_i h_i |e_i\rangle\langle e_i|, \quad (25)$$

and h_i are the singular values of H . S' is of the same form as in (12). We now have

Theorem 3. *The matrix H_n is a generator of a classical stochastic process, if and only if \mathcal{E}_{H_n} is the Lindblad generator.*

Proof. The off-diagonal entries of H_n are all of the form $h \cdot e_i$ (for $i = 1, 2, 3$), where $h := (h_1, h_2, h_3)$ and the diagonal elements are all equal to $h \cdot e_0$. A similar calculation which led to (20), now leads to

$$\mathcal{E}_{H_n}(\rho) = \frac{1}{2} \sum_i h_i \text{tr}(\rho \sigma_i) \sigma_i. \quad (26)$$

We now resort to a theorem in [12] according to which, for a map $\rho \rightarrow L(\rho)$, L is a Lindblad generator if and only if it is Hermitian, $L^*(I) = 0$, where L^* denotes the dual map, and $\omega^\perp \hat{L}^\Gamma \omega^\perp \geq 0$. In this last expression, $\omega^\perp = I - |\omega\rangle\langle\omega|$ where $|\omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a maximally entangled state, and \hat{L} is the linearized form of the map, when it acts on the vectorized form of ρ , $\rho_{i,j} \rightarrow \hat{L}_{ij,kl} \rho_{k,l}$ and finally $(\cdot)^\Gamma$ represents the involution $(\hat{L}^\Gamma)_{ij,kl} = \hat{L}_{ik,jl}$. The first two conditions are obviously satisfied. For the third we note from (26) that $\hat{\mathcal{E}}_{H_n}^\Gamma = \frac{1}{2} \sum_i h_i (\sigma_i \otimes \sigma_i^t)$ that the eigenvalues of $\omega^\perp \hat{\mathcal{E}}_{H_n}^\Gamma \omega^\perp$ are exactly equal to $\frac{1}{2} h \cdot e_i$, which are nothing but the off diagonal elements of H . So \mathcal{E}_{H_n} is a quantum Lindblad generator if and only if H_n is a classical generator. \square

Corollary 1. *For any matrix of the form $H = S' H_n S'^T$, where S' is as in (12), the corresponding \mathcal{E}_H will also be a Lindblad generator.*

Proof. Let U be as before, (i.e. after equation (19)). Then it is clear that

$$\mathcal{E}_H(\rho) = U^{-1} \mathcal{E}_{H_n}(U \rho U^{-1}) U, \quad (27)$$

Using the relation $\hat{\mathcal{E}}^\Gamma = 2\tau(\mathcal{E})$ we find

$$\hat{\mathcal{E}}_H^\Gamma = (U \otimes U^*)^{-1} \hat{\mathcal{E}}_{H_n}^\Gamma (U \otimes U^*). \quad (28)$$

Now since ω^\perp commutes with $U \otimes U^*$, we find $\omega^\perp \hat{\mathcal{E}}_{H_n}^\Gamma \omega^\perp \geq 0$ if and only if $\omega^\perp \hat{\mathcal{E}}_H^\Gamma \omega^\perp \geq 0$. Therefore the six-parameter family of the matrices H also lead to quantum Lindblad generators. In such a case, if $Q = e^H$, then we find that $\mathcal{E}_Q = e^{\mathcal{E}_H}$, that is H and \mathcal{E}_H generate respectively the classical stochastic process and its quantum image. \square

Returning to our map $Q \rightarrow \mathcal{E}_Q$, we would like to elaborate on the decomposition of stochastic matrices and the effect it has on their quantum counterpart. Consider the normal form Q_n . For λ to lie inside the tetrahedron, the magnitude of components λ_i should be less than one. The matrix Q_n shrinks the Bloch Tetrahedron in the principal directions by a factor λ_i . The transformed tetrahedron

still lies inside Δ . However rotations and reflections induced by S and T will bring small parts of the new transformed tetrahedron to outside Δ . The reason is that the matrices S and T are only orthogonal and not stochastic [13]. Therefore while SQ_nT may be non-stochastic, its quantum counterpart $\mathcal{U} \circ \mathcal{E}_{Q_n} \circ \mathcal{V}$ is certainly a qubit channel.

Conversely there may be stochastic matrices for which λ does not belong to Δ , whose quantum image is not a qubit channel. As expected, there shouldn't be a one-to-one correspondence between classical and quantum processes, nevertheless we have found a map from stochastic matrices, whose image has a very large overlap with qubit channels and have also identified why such an overlap is not complete. It seems that two types of mapping are possible, one can either use orthonormal and hence non-positive operators, (like $A_\mu := \frac{1}{2}(I + e_\mu \cdot \sigma)$ in $d = 2$) or else choose positive and non-orthonormal operators, (like $A_\mu := \frac{1}{2}(I + \frac{1}{\sqrt{3}} e_\mu \cdot \sigma)$). We think that these two kinds of maps complement each other, and they are induced from two types of mapping the Bloch tetrahedron to the Bloch sphere, a subject which we will explore in more detail elsewhere [14].

To what extent these considerations can be generalized? Let $\{A_\mu\}$ and $\{B_\mu\}$ be two basis sets of Hermitian operators in M_{d^2} , (the space of $d^2 \times d^2$ matrices) with the following properties:

$$\sum_\mu A_\mu = I, \quad \text{tr}(B_\nu) = 1. \quad (29)$$

We can then define a map Φ , which takes any $Q \in M_{d^2}$ to an \mathcal{E}_Q defined as

$$\mathcal{E}_Q(\rho) := \sum_{\mu,\nu} Q_{\mu,\nu} \text{tr}(\rho A_\nu) B_\mu. \quad (30)$$

If the operators $\{A_\mu\}$ and $\{B_\nu\}$ are positive, (which will no longer be orthonormal), then Φ maps any stochastic (and doubly stochastic) matrix to a CPT (and unital) map, readily verified by forming the Choi-Jamiolkowski matrix of \mathcal{E}_Q . It also follows the composition rule

$$\mathcal{E}_{Q_1} \mathcal{E}_{Q_2} = \mathcal{E}_{Q_1 G Q_2}, \quad (31)$$

where G is the stochastic matrix given by $G_{\mu,\nu} = \text{tr}(A_\mu B_\nu)$.

In summary we have suggested a way for mapping classical d^2 stochastic matrices to quantum channels in d dimensions. In particular, in $d = 2$, we have shown that four dimensional probability vectors and stochastic matrices, can be described in almost complete analogy with qubit density matrices and qubit quantum channels. This similar structure allows us to make a one-to-one correspondence between the normal form of doubly stochastic matrices in four dimensions and the normal form of qubit quantum channels. These findings may

be expanded in a few directions. Besides considering the case of non-doubly stochastic matrices, it obviously begs for a generalization to arbitrary dimensions, a task which is highly non-trivial. The immediate task is to generalize it to the case of multi-qubits, where a uniformly spaced basis like (6) exists due to the existence of Hadamard matrices in 2^k dimensions. Once this is done, one can also see how classical correlations lead to correlated quantum channels. Moreover these findings and the homomorphic property (18), may shed light on the problem of divisibility and Markovianity [5] of classical and quantum maps and relate this problem in these two domains.

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